JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 28, No. 4, July–August 2005

Improved Integer Ambiguity Resolution Technique for Fixed Arrays

Jonathan D. Wolfe* and Jason L. Speyer†
University of California, Los Angeles, Los Angeles, California 90095-1597

A method for quickly and accurately determining the integer ambiguities associated with carrier phase measurements on a rigid antenna array is presented. Residuals generated from the difference between the global positioning system measurements and the hypothesized value have been used in the multiple-hypothesis Wald sequential probability test (MHWSPT) to resolve the integer ambiguities. The new technique involves augmenting these residuals with new residuals constructed from the known baseline distances between the antennas in the array. Experimental results using this augmented set of residuals in the MHWSPT demonstrate an approximately eightfold reduction compared to that using only the residual set without augmentation in the time required to determine the integer ambiguities to desired levels of certainty.

Nomenclature

$b^{(j)}$	=	common mode global positioning system (GPS) code measurement error associated with GPS satellite j
c	=	speed of light in a vacuum
m	=	number of mutually observed GPS satellites
$N_i^{(j)}$	=	ambiguity in phase measurement between GPS satellite j and user i
$n_i^{(j)}$	=	noncommon mode GPS code measurement error between GPS satellite j and antenna i
$\boldsymbol{S}^{(j)}$	=	position of GPS satellite j in inertial space
$\boldsymbol{x}_i$	=	position of ith antenna in inertial space
$\beta^{(j)}$	=	common mode GPS carrier phase measurement error associated with GPS satellite j
$\Delta N^{(j)}$	=	$N_2^{(j)} - N_1^{(j)}$
$\Delta n^{(j)}$	=	$n_2^{(j)} - n_1^{(j)}$
Δt	=	$\Delta t_2 - \Delta t_1$
Δt_i	=	clock error between ith user and GPS time
$\Delta \boldsymbol{x}$	=	$\boldsymbol{x}_2 - \boldsymbol{x}_2$
$\Delta \eta^{(j)}$	=	$\eta_2^{(j)} - \eta_1^{(j)}$
$\eta_i^{(j)}$	=	noncommon mode GPS carrier phase measurement error between GPS satellite j and antenna i
λ	=	carrier signal wavelength
$\tilde{\rho}_i^{(j)}$	=	code GPS pseudomeasurement from GPS satellite j to antenna i
$\nabla \Delta \boldsymbol{N}$	=	vector of integer ambiguities

I. Introduction

FOR control and navigation applications, the attitudes of airplanes, spacecraft, and water vessels have long been measured using inertial measurement devices such as gyroscopes. These instruments require periodic recalibration and are relatively expensive. The widespread adoption of the global positioning system (GPS) over the past two decades has added a new position and velocity sensing instrument to the suite of navigation instruments that is relatively inexpensive and is self-calibrating. With the coming of GPS, the attitude of a vehicle in inertial space can also now be measured in a new way.

The basic GPS technology was originally developed as a tool for determining the position and velocity of a radio-frequency antenna with respect to an Earth-centered coordinate system. Researchers quickly found that they could enhance the accuracy of their measurements by measuring the positions of two or more GPS antennas relative to each other. In particular, to the problem of attitude estimation, researchers realized that by determining the relative positions between GPS antennas that are affixed to a rigid platform, the orientation of the platform can be determined in inertial space. To make accurate estimates of the platform's attitude, the differential GPS measurements must have centimeter accuracy. Although GPS technology has advanced sufficiently to allow differential carrier phase GPS measurements with this accuracy, there are certain problems associated with using these measurements.

Whereas the differential carrier phase between the antennae can be measured accurately and precisely to the centimeter level, the integer numbers of full carrier phase cycles originating from the same GPS satellite that lie between the antennas are not measured. The phase-lock loops tracking the carrier signal do measure the number of cycles that pass once the tracking begins, but the initial values remain unknown. The problem of determining these initial values, called the integer ambiguities, has been studied extensively.[1–7]

When the fixed geometry of an antenna array can be used as a constraint, the process of finding the integer ambiguities can be accelerated. Prior integer resolution methods that exploit this constraint fall into two basic categories. Some authors attempt to use the fixed-baseline constraint to resolve the integers using the measurement data from a single epoch.[8–11] Because these instantaneous methods do not smooth over many measurements, they are vulnerable to sensor noise and may converge to incorrect solutions. As a result, a single noisy measurement can seriously impair the accuracy of the system because an erroneous integer value will typically reduce the accuracy of the relative distance estimate by an order of magnitude.

Another class of integer resolution methods for fixed-geometry arrays uses information about the motion of the GPS satellites and the antenna platform.[12,13] Motion data are collected over a period of time, then the integers are resolved in a batch solution. The use of additional measurements improves the integrity of the solution when compared to the instantaneous methods. Unfortunately, it may require several minutes of data to generate a solution.

This paper extends prior work on integer resolution to the special problem of resolving the integer ambiguities of rigid antenna arrays, so that we can find the attitude of the array.[6] By the use of the extra constraints induced by the array's rigidity, we can reduce the time that it takes to converge on the correct integer ambiguities. As in previous work, our algorithm begins by calculating a solution using data from only a single measurement epoch. As new measurements from later epochs are acquired, the solution algorithm calculates new

---

Received 19 March 2004; revision received 3 December 2004; accepted for publication 3 December 2004.

*Research Engineer, Department of Mechanical and Aerospace Engineering. Member AIAA.

†Professor, Department of Mechanical and Aerospace Engineering. Fellow AIAA.

solutions for the integer ambiguity and estimates of its confidence in these solutions. Our algorithm asymptotically minimizes the expected number of data samples that it must take before announcing a solution, given a desired probability that the solution is correct. (The minimum is asymptotic in the sense that it allows sampling to possibly continue indefinitely.[14–19]) As a result of implementing the length constraint, our experiments show that the new method resolves the integer ambiguity set approximately eight times faster than the standard method used for the unconstrained problem.

This paper is organized as follows: Section II provides a review of the fundamentals of differential GPS. In Sec. III we introduce a residual process that uses information about the known baseline lengths between the antennae in the array. Residuals are generated from the difference between the GPS measurements and hypothesized values of the integer ambiguities. We augment the standard residuals used previously[6] with this new residual, and then analyze the augmented residual set with a procedure that determines the integer ambiguities using a multiple-hypothesis Wald sequential probability test (MHWSPT). In Sec. IV, experimental results from a fixed baseline array are presented. In our experiments, we show that the MHWSPT applied to the augmented residual can resolve the integer ambiguities about eight times more quickly to the correct integer set than when only the standard residuals are used. Section V concludes the paper.

II. Differential GPS Preliminaries

To simplify the exposition, we will confine ourselves to the case of differential GPS between two measurement stations. Extensions to cases with more measurement stations is a straightforward matter. The measurement stations, identified by the subscripts 1 and 2, each collect GPS code and carrier phase pseudomeasurements corresponding to a set of m mutually observed GPS satellites. The code and carrier phase pseudomeasurements collected at station i corresponding to the jth GPS satellite are

$$\tilde{\rho}_i^{(j)} = \left\| \boldsymbol{S}^{(j)} - \boldsymbol{x}_i \right\| + c\Delta t_i + n_i^{(j)} + b^{(j)} \tag{1}$$

$$\lambda\tilde{\phi}_i^{(j)} = -\lambda N_i^{(j)} + \left\| \boldsymbol{S}^{(j)} - \boldsymbol{x}_i \right\| + c\Delta t_i + \eta_i^{(j)} + \beta^{(j)} \tag{2}$$

where $N_i^{(j)}$ is the initial number of full wavelengths between the jth GPS satellite and user i. (After the initial time, increases and decreases in the number of cycles between the user and the satellite are counted by a phase lock loop and reflected in $\tilde{\phi}_i^{(j)}$.) The terms $n_i^{(j)}$, $b^{(j)}$, $\eta_i^{(j)}$, and $\beta^{(j)}$ reflect noise from various sources. The code pseudorange $\tilde{\rho}_i^{(j)}$ and the carrier phase $\tilde{\phi}_i^{(j)}$ are measurements, the carrier wavelength λ is a known constant, and the satellite positions $\{\boldsymbol{S}^{(j)}\}$, $j = 1, 2, \ldots, m$, are assumed to be known. All of the other variables in Eqs. (1) and (2) are unknown.

The term Δt_i represents the unknown bias between the ith user's clock and GPS time. We also include in this clock bias term the receiver oscillator phase difference term defined in Ref. 20. This inclusion is consistent with the analysis in Ref. 21. However, Goad[20] adds the receiver oscillator phase differences into the integer ambiguity $N_i^{(j)}$ instead. When this approach is used, the integer ambiguities will take noninteger values, although the double differencing operation described later in this section will cancel out the noninteger parts of each $N_i^{(j)}$. Note that the receiver oscillator phase difference is common between measurements associated with the same GPS satellite vehicle (SV), just like the clock error. Therefore, these combined errors can be estimated, or eliminated by a double difference.

Let us now assume that the antennae are affixed to a rigid body and that we wish to determine the attitude of that body in an inertial space. To determine the attitude, we do not need to know the absolute positions of the antennas. The relative positions of the antennas in inertial space are sufficient to determine the attitude, without any knowledge of the absolute position of the body.

When we subtract the measurements corresponding to one antenna from those measured at another antenna, we obtain a measurement that is a function of the relative distance between the two antennas,

$$\Delta\tilde{\rho}^{(j)} = \tilde{\rho}_2^{(j)} - \tilde{\rho}_1^{(j)} = \left\| \boldsymbol{S}^{(j)} - \boldsymbol{x}_2 \right\| - \left\| \boldsymbol{S}^{(j)} - \boldsymbol{x}_1 \right\| + c\Delta t + \Delta n^{(j)} \tag{3}$$

$$\lambda\Delta\tilde{\phi}^{(j)} = \lambda\tilde{\phi}_2^{(j)} - \lambda\tilde{\phi}_1^{(j)} = -\lambda\Delta N^{(j)} + \left\| \boldsymbol{S}^{(j)} - \boldsymbol{x}_2 \right\| - \left\| \boldsymbol{S}^{(j)} - \boldsymbol{x}_1 \right\| + c\Delta t + \Delta\eta^{(j)} \tag{4}$$

where $\Delta t \triangleq \Delta t_2 - \Delta t_1$, $\Delta N^{(j)} \triangleq N_2^{(j)} - N_1^{(j)}$, $\Delta n^{(j)} \triangleq n_2^{(j)} - n_1^{(j)}$, and $\Delta\eta^{(j)} \triangleq \eta_2^{(j)} - \eta_1^{(j)}$. Note that the common-mode noises $\{b^{(j)}\}$ and $\{\beta^{(j)}\}$ have been eliminated. Because the common-mode errors in the GPS signal are relatively much larger than the other measurement errors, creating these differential measurements allows us access to measurements with lower noise when the antennas are close (within kilometers) to each other. Note also that if both of the antennas were connected to the same GPS receiver, and the line biases between all of the antennas and the receiver had been calibrated in advance, then the differential clock bias $c\Delta t$ would always be zero.

We will now linearize the measurement equations (3) and (4) about some reference position $\boldsymbol{x}_0$ that is nearby both of our two antennas. (Usually, this is just a poor position estimate of one of the antennas relative to the center of the Earth, constructed using absolute code GPS measurements.) Then the quantity $\|\boldsymbol{S}^{(j)} - \boldsymbol{x}_i\|$ can be approximated to first order as

$$\left\| \boldsymbol{S}^{(j)} - \boldsymbol{x}_i \right\| \approx \left\| \boldsymbol{S}^{(j)} - \boldsymbol{x}_0 \right\| + \boldsymbol{h}^{(j)}(\boldsymbol{x}_i - \boldsymbol{x}_0) \tag{5}$$

where

$$\boldsymbol{h}^{(j)} \triangleq \left. \begin{bmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} & \dfrac{\partial r}{\partial z} \end{bmatrix} \right|_{(x_0, y_0, z_0)}$$

$$r \triangleq \sqrt{\left(X^{(i)} - x_0\right)^2 + \left(Y^{(i)} - y_0\right)^2 + \left(Z^{(i)} - z_0\right)^2}$$

and the coordinates of $\boldsymbol{S}^{(j)}$ and $\boldsymbol{x}_0$ are given by $(X^{(j)}, Y^{(j)}, Z^{(j)})$ and (x_0, y_0, z_0), respectively.

The differential GPS measurement equations (3) and (4) may, thus, be approximated as

$$\Delta\tilde{\rho}^{(j)} = \boldsymbol{h}^{(j)}(\boldsymbol{x}_2 - \boldsymbol{x}_1) + c\Delta t + \Delta n^{(j)} \tag{6}$$

$$\lambda\Delta\tilde{\phi}^{(j)} = -\lambda\Delta N^{(j)} + \boldsymbol{h}^{(j)}(\boldsymbol{x}_2 - \boldsymbol{x}_1) + c\Delta t + \Delta\eta^{(j)} \tag{7}$$

Arranging these equations in a vector form associated with the satellite set in view yields

$$\Delta\tilde{\boldsymbol{\rho}} = \boldsymbol{H}\Delta\boldsymbol{x} + \boldsymbol{U}c\Delta t + \Delta\boldsymbol{n} \tag{8}$$

$$\lambda\Delta\tilde{\boldsymbol{\phi}} = -\lambda\Delta\boldsymbol{N} + \boldsymbol{H}\Delta\boldsymbol{x} + \boldsymbol{U}c\Delta t + \Delta\boldsymbol{\eta} \tag{9}$$

where $\Delta\boldsymbol{x} \triangleq \boldsymbol{x}_2 - \boldsymbol{x}_1$ and $\boldsymbol{U} \triangleq [1 \ 1 \ \cdots \ 1]^T$. Note here that the relative clock error $c\Delta t$ appears in every equation represented inside the matrix equations (8) and (9). Hence, we can multiply these vector equations on the left by an elementwise integer valued left annihilator $\boldsymbol{A}$ of the rank one matrix $\boldsymbol{U}$ and arrive at double differenced measurement equations that do not depend on the clock biases. ($\boldsymbol{A} \in \mathbb{R}^{(m-1)\times(m)}$ is a left annihilator of $\boldsymbol{U}$ when it has full row rank and $\boldsymbol{AU} = \boldsymbol{0}_{(m-1)}$. If $m > 1$, such an integer-valued left annihilator will always exist. One such $\boldsymbol{A}$ corresponds to subtracting the first row of equations from all subsequent rows.) Thus,

$$\nabla\Delta\tilde{\boldsymbol{\rho}} = \nabla\boldsymbol{H}\Delta\boldsymbol{x} + \nabla\Delta\boldsymbol{n} \tag{10}$$

$$\lambda(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) = \nabla\boldsymbol{H}\Delta\boldsymbol{x} + \nabla\Delta\boldsymbol{\eta} \tag{11}$$

where

$$\nabla\Delta\tilde{\boldsymbol{\rho}} = \boldsymbol{A}\Delta\tilde{\boldsymbol{\rho}}, \qquad \nabla\boldsymbol{H} = \boldsymbol{AH}, \qquad \nabla\Delta\boldsymbol{n} = \boldsymbol{A}\Delta\boldsymbol{n} \tag{12}$$

$$\nabla\Delta\tilde{\boldsymbol{\phi}} = \boldsymbol{A}\Delta\tilde{\boldsymbol{\phi}}, \qquad \nabla\Delta\boldsymbol{N} = \boldsymbol{A}\Delta\boldsymbol{N}, \qquad \nabla\Delta\boldsymbol{\eta} = \boldsymbol{A}\Delta\boldsymbol{\eta} \tag{13}$$

We will refer to the operation of subtracting a quantity corresponding to a difference between two antennas as taking a single difference and to the operation of subtracting two single differences corresponding to different SVs as taking a double difference. Note that, after making the initial phase measurement, increases and decreases in the number of cycles between the user and the satellite are counted by a phase-lock loop and reflected in $\{\tilde{\phi}_i^{(j)}\}$; therefore, $\nabla\Delta\boldsymbol{N}$ is an unknown, constant, integer-valued vector.

As we noted before, when every antenna connects to a single receiver, the differential clock bias term $c\Delta t$ equals zero. Therefore, the single differenced measurement equations (8) and (9) reduce to

$$\Delta\tilde{\boldsymbol{\rho}} = \boldsymbol{H}\Delta\boldsymbol{x} + \Delta\boldsymbol{n} \tag{14}$$

$$\lambda(\Delta\tilde{\boldsymbol{\phi}} + \Delta\boldsymbol{N}) = \boldsymbol{H}\Delta\boldsymbol{x} + \Delta\boldsymbol{\eta} \tag{15}$$

If one adds the receiver oscillator phase differences to the integer ambiguities as in Ref. 20, they are canceled in the single difference, and $\Delta\boldsymbol{N}$ is, thus, an integer-valued constant vector. Using the preceding equations instead of their double-differenced equivalents (11) and (10), therefore, allows one to resolve the relative positions using one fewer GPS satellite than the two-receiver case requires. Unfortunately, the experimental results shown later in the paper had to be conducted using two separate receivers because the primary purpose of that experiment was verification of a system where the length of the baseline was undetermined in general. For the rest of the paper, we will, therefore, use only the more generally applicable double-differenced measurement equations, but keep in mind that when single differences are substituted for double differences in the sequel the results will have the same form.

Once we determine the value of the constant, integer-valued vector $\nabla\Delta\boldsymbol{N}$, we can generate very accurate relative position estimates using the carrier phase measurement equation (11). Our objective is, thus, to determine an accurate value for the integer ambiguity $\nabla\Delta\boldsymbol{N}$ as rapidly as possible.

III. Finding Integer Ambiguity When Antennas Are Mounted to Rigid Body

We will now illustrate a method for determining the integer ambiguity $\nabla\Delta\boldsymbol{N}$ that takes advantage of our knowledge of the fixed distance between the antennas.

A. Construction of Residual Based on $\|\Delta\boldsymbol{x}\|$

To determine the probability that a particular hypothesis set for the integer ambiguity is correct, we analyze residual processes associated with the GPS measurements. We define a residual process as a random variable whose statistics depend on the actual hypothesis set. When a residual process is independent in time, statistical hypothesis testing methods can use it to determine the probability that a given hypothesis is correct to an arbitrary degree of certainty.[14–19]

Suppose now that we know the distance $d \triangleq \|\Delta\boldsymbol{x}\|$ between two antennas, but not their relative orientation. We will use this information to construct a new measurement residual that, when combined with the measurement residuals already in use, will allow the hypothesis tests to converge more rapidly than with the standard residual alone.

We begin with the linearized vector form of the double-differenced carrier phase differential GPS measurements of Eq. (11),

$$\lambda(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) = \nabla\boldsymbol{H}\Delta\boldsymbol{x} + \nabla\Delta\boldsymbol{\eta} \tag{16}$$

If $\nabla\boldsymbol{H}$ has full column rank, then there exists a left pseudoinverse $(\nabla\boldsymbol{H})^\dagger$ so that

$$(\nabla\boldsymbol{H})^\dagger\nabla\boldsymbol{H} = \boldsymbol{I} \tag{17}$$

Therefore, by multiplying Eq. (16) by $(\nabla\boldsymbol{H})^\dagger$ on the left and rearranging terms, we obtain an expression for $\Delta\boldsymbol{x}$:

$$\Delta\boldsymbol{x} = (\nabla\boldsymbol{H})^\dagger[\lambda(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) - \nabla\Delta\boldsymbol{\eta}] \tag{18}$$

Note that except for the error associated with constructing the linearized equation (16), this expression for $\Delta\boldsymbol{x}$ is exact.

We now can substitute this expression into the definition of the known quantity d:

$$d^2 = (\Delta\boldsymbol{x})^T\Delta\boldsymbol{x} \tag{19}$$

$$\begin{aligned} &= \lambda^2(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) \\ &\quad - 2\lambda(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger\nabla\Delta\boldsymbol{\eta} \\ &\quad + (\nabla\Delta\boldsymbol{\eta})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger\nabla\Delta\boldsymbol{\eta} \end{aligned} \tag{20}$$

$$\begin{aligned} &\approx \lambda^2(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) \\ &\quad - 2\lambda(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger\nabla\Delta\boldsymbol{\eta} \end{aligned} \tag{21}$$

where Eq. (21) approximates Eq. (20) to first order by assuming small values of $\nabla\Delta\boldsymbol{\eta}$.

Rearranging Eq. (21) yields a new residual for hypothesis testing,

$$\begin{aligned} R_d &\triangleq d^2 - \lambda^2(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) \\ &\approx -2\lambda(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger\nabla\Delta\boldsymbol{\eta} \end{aligned} \tag{22}$$

Note that this same residual was used in Ref. 8, but there it was used solely for statistical tests that lasted only for one measurement epoch. Here, we will use this residual as part of a multi-time-step integer resolution scheme. Note also that the statistics of R_d are related to those of the residuals used in Ref. 6,

$$\boldsymbol{R}_g = \lambda\boldsymbol{E}(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) = \boldsymbol{E}\nabla\Delta\boldsymbol{\eta} \tag{23}$$

$$\boldsymbol{R}_s = \lambda(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) - \nabla\Delta\tilde{\boldsymbol{\rho}} = \nabla\Delta\boldsymbol{\eta} - \nabla\Delta\boldsymbol{n} \tag{24}$$

where $\boldsymbol{E}$ is a matrix annihilator such that

$$\boldsymbol{E}\nabla\boldsymbol{H} = \boldsymbol{0}_{(m-3)}, \qquad \boldsymbol{E}\boldsymbol{E}^T = \boldsymbol{I}_{(m-3)\times(m-3)}$$

Such an unitary matrix annihilator will always exist if $\nabla\boldsymbol{H}$ has full column rank. The residuals $\boldsymbol{R}_g$ and $\boldsymbol{R}_s$ are, thus, random variables that are multiples of the random variables $\nabla\Delta\boldsymbol{\eta}$ and $\nabla\Delta\boldsymbol{n}$. Note that Eq. (23) is obtained from Eq. (11) by using $\boldsymbol{E}$ to annihilate $\nabla\boldsymbol{H}$, and Eq. (24) is obtained by simply subtracting Eq. (10) from Eq. (11). Combining the three residuals, we have the random process

$$\begin{aligned} \boldsymbol{r} &\triangleq \begin{bmatrix} \boldsymbol{R}_s \\ \boldsymbol{R}_g \\ R_d \end{bmatrix} \\ &= \begin{bmatrix} \lambda(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) - \nabla\Delta\tilde{\boldsymbol{\rho}} \\ \lambda\boldsymbol{E}(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) \\ d^2 - \lambda^2(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) \end{bmatrix} \\ &= \begin{bmatrix} -\boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{E} \\ \boldsymbol{0} & -2\lambda(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger \end{bmatrix} \begin{bmatrix} \nabla\Delta\boldsymbol{n} \\ \nabla\Delta\boldsymbol{\eta} \end{bmatrix} \end{aligned} \tag{25}$$

If $\nabla\Delta\boldsymbol{n}$ and $\nabla\Delta\boldsymbol{\eta}$ are random processes whose elements are independent in time, then the sequence $\{\boldsymbol{r}(k)\}$ is suitable for statistical hypothesis testing.

In practice, the antenna array may not be perfectly rigid. The fixed-baseline part of the residual $\boldsymbol{r}$ may be modified to account for the flexibility of the array by introducing a noise term to the baseline distance d in Eq. (20). However, doing this might require accurate modeling and estimation of the structural dynamics of the array, significantly increasing the complexity of the problem.

B. Determining Integer Ambiguity Using the Augmented Residual

We may now determine the correct set of integer biases $\{\nabla\Delta N^{(i)}\}$, $i = 1, 2, \ldots (m-1)$, using the technique that was developed in Ref. 6. If we assume that $m \geq 4$ and that $\boldsymbol{n}$ and $\boldsymbol{\eta}$ are random vectors with known probability density functions whose elements are independent between time steps, the following technique will resolve the initial integer biases:

Algorithm (MHWSPT)

1) Choose a threshold value T for the probability that a particular set of integers is the correct one. This is the stopping criterion for the test.

2) Generate floating-point valued estimates of $\{\nabla\Delta N^{(i)}\}$, $i = 1, 2, \ldots, (m-1)$, with Eqs. (10) and (11) using standard least-squares techniques. Note that if we attempt to include the length constraint in generating these floating-point valued estimates, the estimation problem becomes nonlinear. To avoid this additional complexity, do not use the length constraint in calculating the floating-point estimates.

3) Choose a collection $\{\mathcal{H}_j\}$, $j = 1, 2, \ldots, n$, of sets of possible integer-valued hypotheses that are near to the floating-point estimates. Developing efficient methods for choosing this set when the noises are Gaussian has been the subject of recent research.[4,5,7]

4) At each time step k, for every hypothesis $\mathcal{H}_j$, evaluate the residual $\boldsymbol{r}$ in Eq. (25) assuming that the value of $\nabla\Delta\boldsymbol{N}$ is given by $\nabla\Delta\boldsymbol{N}_j$. We will denote the residual corresponding to hypothesis $\mathcal{H}_j$ by $\boldsymbol{r}_j(k)$.

5) For each integer set hypothesis $\mathcal{H}_j$ under consideration, define $F_j(k)$ as the probability that hypothesis set j is the correct one, given the time history of all of the residuals $\{\boldsymbol{r}_1(l), \boldsymbol{r}_2(l), \ldots, \boldsymbol{r}_n(l)\}$, $l = 0, 1, 2, \ldots, k$, that is,

$$F_j = P(\mathcal{H}_j | \{\boldsymbol{r}_1(l), \boldsymbol{r}_2(l), \ldots, \boldsymbol{r}_n(l)\}, l = 0, 1, 2, \ldots, k)$$

6) Set each of the initial values $\{F_j(0)\}$. One possibility is to choose the equiprobable distribution $F_j(0) = 1/n$, $j = 1, 2, \ldots, n$, where n is the number of hypothesis sets under consideration.

7) Propagate $\{F_j(k)\}$ using the updates

$$F_j(k+1) = \frac{F_j(k) \cdot f_j[\boldsymbol{r}_j(k+1)]}{\sum_{l=1}^{n} F_l(k) \cdot f_l[\boldsymbol{r}_l(k+1)]}$$

where $f_j[\boldsymbol{r}_j(k+1)]$ indicates the probability density function of $\boldsymbol{r}_j(k+1)$, assuming that hypothesis $\mathcal{H}_j$ is correct. When one of $\{F_j(k)\}$ exceeds the threshold T, stop and declare the corresponding hypothesis set to be the correct one.

For any given probability of the chosen hypothesis being correct, a value for T can be found that ensures that the declared hypothesis is correct with that probability. If the algorithm is allowed in principle to continue indefinitely, it minimizes the expected number of steps before a hypothesis choice is announced.[14–19]

We note that the residual $\boldsymbol{r}$ could also be used to detect the failure of the GPS phase-lock loops to register the passing of a full carrier cycle after the integer ambiguities have been determined. (GPS literature refers to these failures as cycle slips.) Doing this would involve applying the multiple-hypothesis Shiryayev sequential probability test to $\boldsymbol{r}$ in a similar way to the method presented in Ref. 6.

C. Special Case: Gaussian White Noise

Suppose now that the noise sequence $\nabla\Delta\boldsymbol{\eta}$ is a white noise process with zero-mean Gaussian statistics. Then when the value of $\nabla\Delta\boldsymbol{N}$ in Eq. (25) is correct, R_d should also be a zero-mean Gaussian white noise process, with covariance

$$E\left[R_d^2\right] = 4\lambda^2(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger\boldsymbol{V}_{\text{car}}[(\nabla\boldsymbol{H})^\dagger]^T \times (\nabla\boldsymbol{H})^\dagger(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N}) \tag{26}$$

where

$$\boldsymbol{V}_{\text{car}} \triangleq E[\nabla\Delta\boldsymbol{\eta}\nabla\Delta\boldsymbol{\eta}^T] \tag{27}$$

At every measurement epoch, we can construct the residual $\boldsymbol{r}$ using Eq. (25), where $\boldsymbol{r}$ is zero mean with variance

$$\boldsymbol{V} \triangleq \boldsymbol{C}\begin{bmatrix}\boldsymbol{V}_{\text{code}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{V}_{\text{car}}\end{bmatrix}\boldsymbol{C}^T \tag{28}$$

where

$$\boldsymbol{C} \triangleq \begin{bmatrix} -\boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{E} \\ \boldsymbol{0} & -2\lambda(\nabla\Delta\tilde{\boldsymbol{\phi}} + \nabla\Delta\boldsymbol{N})^T[(\nabla\boldsymbol{H})^\dagger]^T(\nabla\boldsymbol{H})^\dagger \end{bmatrix}$$

$$\boldsymbol{V}_{\text{code}} \triangleq E[(\nabla\Delta\boldsymbol{n})(\nabla\Delta\boldsymbol{n})^T], \qquad \boldsymbol{V}_{\text{car}} \triangleq E[(\nabla\Delta\boldsymbol{\eta})(\nabla\Delta\boldsymbol{\eta})^T]$$

When one uses a single receiver, the covariance matrices $\boldsymbol{V}_{\text{code}}$ and $\boldsymbol{V}_{\text{car}}$ are usually diagonal. Note that, in the two-receiver case, the double differencing operation fully populates the matrices $\boldsymbol{V}_{\text{code}}$ and $\boldsymbol{V}_{\text{car}}$.

Note now that the residual $\boldsymbol{r}$ is a function of the choice of the integer hypotheses $\{\nabla\Delta N_i\}$, $i = 1, 2, \ldots, n$. At every time k, each hypothesis of the integer ambiguity $\nabla\Delta N_i(k)$ corresponds to a different possible value of the residual and the noise covariance, denoted $\boldsymbol{r}_i(k)$ and $\boldsymbol{V}_i(k)$. Hence, for any hypothesis i at time k, we can calculate the corresponding probability density function $f_j[\boldsymbol{r}_i(k)]$:

$$f_j[\boldsymbol{r}_i(k)] = \left[1\big/(2\pi)^{a(k)/2}|\boldsymbol{V}_i(k)|^{a(k)/2}\right] \times \exp\left\{-0.5\boldsymbol{r}_i(k)^T[\boldsymbol{V}_i(k)]^{-1}\boldsymbol{r}_i(k)\right\} \tag{29}$$

where $a(k)$ is the dimension of the residual vector $\boldsymbol{r}_i(k)$.

The Gaussian nature of the measurements simplifies the propagation of the conditional probabilities used in the MHWSPT:

$$\begin{aligned}
F_j(k+1) &= \frac{F_j(k) \cdot f_j[\boldsymbol{r}_j(k+1)]}{\sum_{l=1}^{n} F_l(k) \cdot f_l[\boldsymbol{r}_l(k+1)]} = \frac{F_j(k)\left[1\big/(2\pi)^{a(k+1)/2}|\boldsymbol{V}_j(k+1)|^{a(k+1)/2}\right]\exp\left\{-0.5\boldsymbol{r}_j(k+1)^T[\boldsymbol{V}_j(k+1)]^{-1}\boldsymbol{r}_j(k+1)\right\}}{\sum_{l=1}^{n} F_l(k)\left[1\big/(2\pi)^{a(k+1)/2}|\boldsymbol{V}_l(k+1)|^{a(k+1)/2}\right]\exp\left\{-0.5\boldsymbol{r}_l(k+1)^T[\boldsymbol{V}_l(k+1)]^{-1}\boldsymbol{r}_l(k+1)\right\}} \\
&= S(k+1)F_j(k)\frac{1}{(2\pi)^{a(k+1)/2}|\boldsymbol{V}_j(k+1)|^{a(k+1)/2}}\exp\left\{-0.5\boldsymbol{r}_j(k+1)^T[\boldsymbol{V}_j(k+1)]^{-1}\boldsymbol{r}_j(k+1)\right\} \\
&= S(k+1)\,S(k)\frac{1}{(2\pi)^{a(k+1)/2}|\boldsymbol{V}_j(k+1)|^{a(k+1)/2}}\frac{1}{\left(2\pi^{a(k)/2}\right)|\boldsymbol{V}_j(k)|^{a(k)/2}} \\
&\quad \times \exp\left\{-0.5\boldsymbol{r}_j(k+1)^T[\boldsymbol{V}_j(k+1)]^{-1}\boldsymbol{r}_j(k+1) - 0.5\boldsymbol{r}_j(k)^T[\boldsymbol{V}_j(k)]^{-1}\boldsymbol{r}_j(k)\right\} \\
&= \left[\prod_{l=1}^{k+1} S(l)\right]\left[\prod_{l=1}^{k+1}\frac{1}{(2\pi)^{a(l)/2}|\boldsymbol{V}_j(l)|^{a(l)/2}}\right]\exp\left\{-\sum_{l=1}^{k+1}0.5\boldsymbol{r}_j(l)^T[\boldsymbol{V}_j(l)]^{-1}\boldsymbol{r}_j(l)\right\}
\end{aligned} \tag{30}$$

where we define

$$S(k+1) \triangleq 1 \Bigg/ \left(\sum_{l=1}^{n} F_l(k) \frac{1}{(2\pi)^{a(k+1)/2} |\boldsymbol{V}_l(k+1)|^{a(k+1)/2}} \times \exp\left\{ -0.5 \boldsymbol{r}_l(k+1)^T \boldsymbol{V}_l(k+1)^{-1} \boldsymbol{r}_l(k+1) \right\} \right) \tag{31}$$

We can, therefore, see that the exponential part of the probability is a weighted sum of the squares of the residuals, exactly the same as the chi-squared cost function used in conventional GPS integer ambiguity resolution. However, other terms also appear in the conditional probability $F_j(k+1)$ that do not appear in a conventional sum-of-squares cost function.

Most important, the probability F_j is created by normalizing by S, the effect that the new information has on the other hypotheses. Hence, the importance of the new information about the jth hypothesis is relative to the importance of the new information's effect on the probabilities of other hypotheses being correct. Unlike a conventional chi-squared test that checks against an alternative hypothesis that is a Gaussian random variable with a known covariance and an unknown mean, whose elements are real numbers, the test that we use knows that the null hypothesis and the set of all alternative hypotheses are unique members of a finite set.

Another way of viewing the difference between our approach and conventional approaches is to consider the conditional probability density function

$$f[\mathcal{H}_i | \boldsymbol{r}_i(1), \boldsymbol{r}_i(2), \ldots, \boldsymbol{r}_i(k)] = \left[\prod_{l=1}^{k+1} \frac{1}{(2\pi)^{a(l)/2} |\boldsymbol{V}_i(l)|^{a(l)/2}} \right] \times \exp\left\{ -\sum_{l=1}^{k+1} 0.5 \boldsymbol{r}_i(l)^T [\boldsymbol{V}_i(l)]^{-1} \boldsymbol{r}_i(l) \right\}$$

We can, therefore, think of the sum

$$\sum_{l=1}^{k+1} \boldsymbol{r}_i(l)^T [\boldsymbol{V}_i(l)]^{-1} \boldsymbol{r}_i(l)$$

as a proxy for the conditional probability density function $f[\mathcal{H}_i | \boldsymbol{r}_i(1), \boldsymbol{r}_i(2), \ldots, \boldsymbol{r}_i(k)]$ and make decisions based on whether this proxy indicates that hypothesis i is probable. However, to construct conditional probabilities for a binary test, we must also know a distribution for the alternative. The most commonly used statistical tests assume an alternative hypothesis that is Gauss normal with an unknown, real-valued mean and the same covariance as the primary hypothesis; but we know that the circumstances are more constrained. There are only a finite number of possible values that the mean can take. Therefore, our scheme differs from the conventional one in that it constrains the probabilities by imposing additional relevant information about the problem.

Our approach to determining the integers also enables us to make easy interpretations of intermediate results before a winning hypothesis emerges. At any time before the declaration of a winning hypothesis, a designer can tell how probable each hypothesis choice is, a very useful diagnostic tool for tuning parameters such as the cutoff probabilities.

Two noteworthy features of the MHWSPT are its sequential nature and its optimality. Because of the sequential nature of the MHWSPT, only the current conditional probabilities $\{F_j(k)\}$, $j = 1, 2, \ldots, n$, need be stored. Another advantage of the MHWSPT is that the MHWSPT has been proven to minimize asymptotically the expected duration of the test, given an acceptable probability of announcing an incorrect choice that is set by the user and the possibility that sampling can continue indefinitely.[14–19]

Our technique for integer ambiguity resolution differs from the conventional techniques in one last important way. Whereas we assumed in this discussion that the measurement noise conformed to Gaussian white noise, the noise in some GPS signals is rarely so. For instance, errors introduced by multipath in GPS code measurements generally follow Rayleigh or Nakagami–Rice distributions. Whereas with sum-of-squares techniques we can only approximate this noise with Gaussian noise, we can introduce the correct probability density functions for the noises into an MHWSPT.

IV. Experimental Results

The effectiveness of the new integer resolution technique was evaluated using a data set that was generated with a test rig consisting of two GPS antennae (Sensor Systems Model S67-1575-96) rigidly mounted a fixed distance across from each other (2.3 m). On Friday, 13 October 2000 at 19:22:33.5 Pacific Standard Time, the test rig was mounted on top of a car across its center axis (like a police light bar). The car was driven for several minutes back and forth on a road in Los Angeles, California (34° 4′ 5″ N, 118° 26′ 53″ W), that ran roughly NNW–SSE while two GPS receivers (Ashtech Model Z-12) collocated with the two antennas recorded L1 and L2 GPS measurements at a rate of 2 Hz. Some portions of the route that the car traversed were straight, whereas in other portions the car turned sharply, and sometimes the car remained stationary. The data set, thus, had measurements taken during periods of high and low receiver position dynamics. Because these data were collected for the purpose of calibrating and testing a differential positioning system and not for measuring attitude, two separate receivers were used in a situation where a single receiver would have performed better. If both of the antennas had been connected to the same receiver and the line delays had been calibrated, then the single-differenced measurements would have contained no differential clock errors, and the double-differencing operation that eliminates differential clock biases would have been unnecessary. The resulting single-differenced data set would have had a smaller noise standard deviation than the one that we used by a factor of $\sqrt{2}$, and the differential positions could have been estimated with as few as three visible GPS satellites, instead of the four visible satellites that were required because we used separate receivers.

To compare the integer resolution technique presented in this paper to one that does not exploit the fixed baseline constraint, we broke the 495-s long data sequence into 395 overlapping 50-s-long data sequences, each one beginning 0.5 s after the previous one and ending 0.5 s later than the previous one. The results from the integer resolution and verification technique presented in this paper were then calculated to compare with results that used the standard residuals. All of the integer resolution techniques assumed that the pseudomeasurement noise sequences were independent, identically distributed random sequences with zero-mean Gaussian statistics. The standard deviations for each L1 code pseudomeasurement were assumed to be 2 m, and the standard deviations for each widelaned carrier phase pseudomeasurement were assumed to be 4 cm. Each integer resolution technique attempted to use information from six satellites. (Hence, there were five initial double-differenced integer ambiguities to be resolved.) Given the widelane wavelength of 86.19 cm and the antenna separation distance of 2.3 m, there can be at most a separation of seven full waves between the two antennas. Therefore, for each double-differenced pseudomeasurement sequence, seven different hypotheses for its initial integer bias were considered. Hence, a total of $7^5 = 16{,}807$ different hypotheses for the initial integer bias set were under consideration, encapsulating all possible values for the integers in a hypercube.

Note that even using widelane measurements requires a large number of hypotheses if we want to account for every possible combination of the integer ambiguities. The situation becomes almost computationally impossible with a single-frequency L1 receiver because the L1 wavelength is much shorter than that of the widelane combination. If we were to contemplate resolving integers in this case, we would have to winnow out the set of possible hypotheses by only considering the more likely ones. Although we could do this, we would run the risk of throwing away the true hypothesis, a dangerous possibility indeed. All search methods for finding integer ambiguities possess this vulnerability, a definite disadvantage compared to motion-based methods.

Because the test platform was not instrumented with calibrated inertial instrumentation, the success of the integer ambiguity searches can only be determined indirectly. Because the road where the test was performed had a shallow grade, we can check whether the relative altitude between the GPS antennas was close to zero. We are fairly certain that the platform did not roll more than 10 deg during the test, and so we should expect all differential altitude measurements to be smaller in magnitude than $2.3\sin(10 \text{ deg}) = 0.3994$ m. When the convergence threshold T was set to 0.999, the sequence of altitude differences calculated at every converged trial had a mean of 0.0180 m and a standard deviation of 0.2017 m, well within the range that we would expect. (For the trials matching a $T = 0.99$ threshold, the corresponding values were a mean of 0.0182 m and a standard deviation of 0.2358 m.)

We can also check whether the integer ambiguity hypotheses that we determined were consistent with the observed trajectory of the vehicle. The vehicle made a 180-deg turn every time it reached the end of the course, and so these turns should appear as 180-deg changes in the estimated yaw angle. Inspecting the estimated yaw angle history does in fact show that the estimated yaw angle would remain constant for long periods, then rapidly change by 180 deg.

We tested our results in one more way by estimating the distance between the two antennas. Because this baseline distance is fixed at 2.3 m, the choice of integer ambiguities when combined with the carrier phase measurements should lead to distance estimates of 2.3 m. When the statistical testing threshold value was set to $T = 0.999$, the estimated distance between the two antennas at all converged trials had a mean of 2.32 m and a standard deviation of 2.46 cm. (For a threshold of $T = 0.99$, the estimated distances for converged trials had a mean of 2.3294 m and a standard deviation of 0.0241 m.) We, therefore, appear to be making our estimates with centimeter-level consistency.

Table 1 compares the number of Wald tests that converged (to a threshold of either 0.99 or 0.999) within 50 s. It also lists the average time that those tests that converged within 50 s tests took to converge. The tests with the augmented residual appear to converge about eight times faster than those using the standard residual. Figure 1 shows the time history of the probability of the most probable hypothesis for a typical trial. Note that the curve representing the use of the new residual is much steeper than the one without augmentation.

The same statistics are presented in Table 1 for 100 adjacent overlapping 50-s long measurement periods that occur in regions corresponding to high or low dynamic behavior of the test rig. It appears from the data in Table 1 that the hypothesis tests had greater difficulty converging to a set of integer hypotheses during the periods of high dynamics. This appears contrary to our expectations because more motion generally improves the observability of the ambiguities. We think that this anomaly stems from two reasons. First, the phase-lock loops in the receivers find it more difficult to track the GPS signals tightly during periods when the platform experiences higher accelerations. Second, in the more dynamic periods, the GPS SVs enter and leave the view of the antennas more frequently. This disrupts the integer resolution process because the uncertainty associated with the integer ambiguities corresponding to the entering satellites forces all of the conditional probabilities lower. We end the section by noting that these effects affect the tests that do not exploit the baseline length constraint far more than those that do use the constraint, again probably because the constrained tests are more likely to decide the integer ambiguities before a disruption can occur.

Table 1 Comparison of convergence success and time to converge and between constrained and unconstrained trials

Length constraint	No. of convergence failures	Average convergence time, s
Convergence threshold of 0.99 for entire set		
With	23	2.0000
Without	235	16.2970
Convergence threshold of 0.999 for entire set		
With	37	2.4566
Without	317	20.5320
Convergence threshold of 0.99, 50-s interval, high dynamics		
With	15	4.1353
Without	52	21.0310
Convergence threshold of 0.999, 50-s interval, high dynamics		
With	20	5.0438
Without	95	48.3000
Convergence threshold of 0.99, 50-s interval, low dynamics		
With	0	1.2150
Without	0	11.6050
Convergence threshold of 0.999, 50-s interval, low dynamics		
With	0	1.5050
Without	0	18.3800

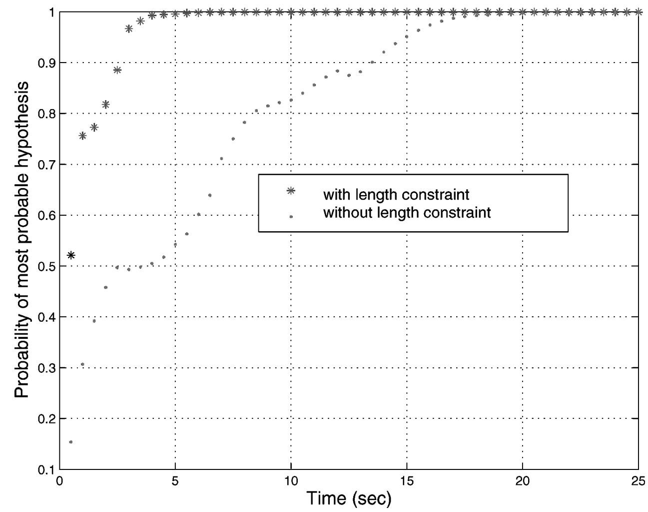

Fig. 1 Comparison of convergence times with and without fixed length constraint.

V. Conclusions

In this paper, we presented a measurement residual for GPS antenna arrays where the baseline lengths between the antennas in the array are fixed and known. One can use this type of antenna array to determine the attitude of a rigid platform in an inertial space using carrier phase GPS. This sort of system can be used as a replacement for or as an adjuvant to conventional navigation instruments such as gyroscopes. To determine the attitude of such an array, the initial number of carrier phase wavelengths between each pair of antennas (the integer ambiguities) must be found. We demonstrated how to process the measurement residual so that the expected time that it takes to determine the integer ambiguities in the carrier phase measurement is minimized for the class of sequential tests where an arbitrary number of samples are available. We also discussed the relationship between our statistical estimation procedures and the conventional chi-squared test and showed that our procedure conditions the estimated probabilities that it creates on more of the problem structure than is used by the chi-squared tests. Our experiments demonstrate that using the fixed-baseline residual dramatically reduces the number of measurement epochs needed to determine the integer ambiguities by approximately eightfold compared to the number of epochs needed using an unconstrained residual.

Acknowledgments

This research was supported in part by NASA Goddard Space Flight Center under Grant NAG5-11384. We thank Walton R. Williamson and Mamoun F. Abdel-Hafez for collecting the experimental GPS data that we used.

References

[1]Counselman, C. C., and Gourevitch, S. A., "Miniature Interferometer Terminals for Earth Surveying: Ambiguity and Multipath with Global Positioning System," *IEEE Transactions on Geoscience and Remote Sensing*, Vol. GE-19, No. 4, 1981, pp. 244–252.

[2]Corbett, S. J., and Cross, P. A., "GPS Single Epoch Resolution," *Survey Review*, Vol. 33, No. 257, 1995, pp. 149–160.

[3]Al-Haifi, Y. M., Corbett, S. J., and Cross, P. A., "Performance Evaluation of GPS Single-Epoch On-the-Fly Ambiguity Resolution," *Navigation: Journal of the Institute of Navigation*, Vol. 44, No. 4, 1997–1998, pp. 479–487.

[4]Teunissen, P. J. G., "A New Method for Fast Carrier Phase Ambiguity Estimation," *IEEE 1994 Position Location and Navigation Symposium*, Inst. of Electrical and Electronics Engineers, Las Vegas, NV, 1994, pp. 562–573.

[5]Teunissen, P. J. G., "The Least-Squares Ambiguity Decorrelation Adjustment: A Method for Fast GPS Integer Ambiguity Estimation," *Journal of Geodesy*, Vol. 70, No. 1–2, 1995, pp. 65–82.

[6]Wolfe, J. D., Williamson, W. R., and Speyer, J. L., "Hypothesis Testing for Resolving Integer Ambiguity in GPS," *Navigation: Journal of the Institute of Navigation*, Vol. 50, No. 1, 2003, pp. 45–56.

[7]Xu, P., "Random Simulation and GPS Decorrelation," *Journal of Geodesy*, Vol. 75, No. 7, 2001, pp. 408–423.

[8]Brown, R. A., "Instantaneous GPS Attitude Determination," *IEEE Aerospace and Electronics Magazine*, Vol. 7, No. 6, 1992, pp. 3–8.

[9]Quinn, P. G., "Instantaneous GPS Attitude Determination," *Proceedings of ION GPS-93*, Inst. of Navigation, Salt Lake City, UT, 1993, pp. 603–615.

[10]Knight, D., "A New Method of Instantaneous Ambiguity Resolution," *Proceedings of ION GPS-94*, Inst. of Navigation, Salt Lake City, UT, 1994, pp. 707–716.

[11]Hazel, G., and Smith, S., "Instantaneous GPS Attitude Tests: Cycle Ambiguity Resolved," *Proceedings of ION GPS-94*, Inst. of Navigation, Salt Lake City, UT, 1994, pp. 683–688.

[12]Cohen, C. E., "Attitude Determination Using GPS," Ph.D. Dissertation, Dept. of Aeronautics and Astronautics, Stanford Univ., Stanford, CA, Dec. 1992.

[13]Cohen, C. E., "Attitude Determination," *Global Positioning System: Theory and Applications*, edited by B. W. Parkinson and J. J. Spilker Jr., Vol. 2, AIAA, Reston, VA, 1996, Chap. 19.

[14]Wald, A., *Sequential Analysis*, Wiley Mathematical Statistics Series, Wiley, New York, 1947.

[15]Bertsekas, D. P., *Dynamic Programming: Deterministic and Stochastic Models*, Prentice–Hall, Englewood Cliffs, NJ, 1987, pp. 132–137, 245–251.

[16]Malladi, D. P., and Speyer, J. L., "A Generalized Shiryayev Sequential Probability Ratio Test for Change Detection and Isolation," *IEEE Transactions on Automatic Control*, Vol. 44, No. 8, 1999, pp. 1522–1534.

[17]Verdenskaya, N. V., and Tartakovskii, A. G., "Asymptotically Optimal Sequential Testing of Multiple Hypotheses for Nonhomogeneous Gaussian Processes in an Asymmetric Situation," *Theory of Probability and Its Applications*, Vol. 36, No. 3, 1991, pp. 536–547.

[18]Tartakovsky, A. G., "Multihypothesis Invariant Sequential Tests with Applications to the Quickest Detection of Signals in Multiple-Resolution-Element Systems," *Proceedings of the 1999 Conference on Information Science and Systems*, Dept. of Electrical Engineering, Princeton Univ., Princeton, NJ, 1998, pp. 906–911.

[19]Draglin, V. P., Tartakovsky, A. G., and Veeravalli, V. V., "Multihypothesis Sequential Probability Ratio Tests—Part I: Asymptotic Optimality," *IEEE Transactions on Information Theory*, Vol. 45, No. 7, 1999, pp. 2448–2461.

[20]Goad, C., "Surveying with the Global Positioning System," *Global Positioning System: Theory and Applications*, edited by B. W. Parkinson and J. J. Spilker Jr., Vol. 2, AIAA, Reston, VA, 1996, Chap. 18.

[21]Leick, A., *GPS Satellite Surveying*, Wiley, New York, 1995, Chap. 8, pp. 255–261.